

# Nonparametric Estimation of the Leverage Effect: A Trade-Off Between Robustness and Efficiency

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## ABSTRACT

We consider two new approaches to nonparametric estimation of the leverage effect. The first approach uses stock prices alone. The second approach uses the data on stock prices as well as a certain volatility instrument, such as the Chicago Board Options Exchange (CBOE) volatility index (VIX) or the Black–Scholes implied volatility. The theoretical justification for the instrument-based estimator relies on a certain invariance property, which can be exploited when high-frequency data are available. The price-only estimator is more robust since it is valid under weaker assumptions. However, in the presence of a valid volatility instrument, the price-only estimator is inefficient as the instrument-based estimator has a faster rate of convergence. We consider an empirical application, in which we study the relationship between the leverage effect and the debt-to-equity ratio, credit risk, and illiquidity. Supplementary materials for this article are available online.

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## 1. Introduction

One of the most important empirical stylized facts about the volatility is the leverage effect, which refers to the generally negative correlation between an asset return and its volatility changes. The term “leverage” dates back to an early influential economic hypothesis of Black (1976) that explains this correlation using the debt-to-equity ratio, a common financial leverage measure. The estimation of the leverage effect is challenging because volatility is not observable.

We develop two qualitatively different approaches to nonparametric estimation of the leverage effect using high-frequency data. We also study the empirical relationship between the leverage effect and the debt-to-equity ratio. Our results extend the large body of research that has used high-frequency data fruitfully to estimate volatility measures of asset returns, see, for example, Andersen et al. (2003). These methods are now commonly used in economics and finance, see, for example, Patton and Verardo (2012), Bandi and Renò (2015), Bollerslev, Li, and Todorov (2016), and Segal, Shaliastovich, and Yaron (2015).

Our first approach to the leverage effect estimation only uses observations on the asset prices. It is an analog of the low-frequency approach that has been common since Black (1976): one first conducts preliminary estimation of the volatility over small windows, and then computes the correlation between returns and the increments of the estimated volatility. However, in general there is an errors-in-variables bias associated with the preliminary estimation of volatility. We propose an estimator that corrects the biases due to the preestimation of volatility, and is valid for a very general class of semimartingales. We call this

estimator the Price-only Realized Leverage (PRL). The rate of convergence of the PRL estimator is  $n^{1/4}$ .

Our second approach is to replace the preliminary estimation of volatility with high-frequency observations of certain financial derivatives. It is well known that financial derivatives contain useful information about volatility. However, the implied volatilities from these derivatives are at best a proxy for the actual volatility as they are contaminated by risk premia. To purge the impact of risk premia, additional assumptions are necessary to model this contamination, which link the risk-neutral dynamics with the objective dynamics. For this purpose, we provide one such condition, that is, Assumption 2, which allows the estimation of the leverage effect using high-frequency data on certain volatility instruments, such as the Chicago Board Options Exchange (CBOE) volatility index (VIX) or Black–Scholes implied volatility. (Since September 22, 2003, the VIX has been constructed by the CBOE using a weighted portfolio of options:  $(VIX_t/100)^2 = \frac{2e^{r_t \tau}}{\tau} (\int_0^{e^{f_{t,\tau}}} \frac{P(\tau, x)}{x^2} dx + \int_{e^{f_{t,\tau}}}^\infty \frac{C(\tau, x)}{x^2} dx)$ , where  $P(\tau, x)$  and  $C(\tau, x)$  are put and call options with time-to-maturity  $\tau$  and strike  $x$ , and  $f_{t,\tau}$  is the log price of forward contracts, see, for example, Carr and Wu 2009.) We call the resulting estimator the instrument-based realized leverage (IRL). We develop the asymptotic theory for this estimator, and show that it has a faster ( $n^{1/2}$ ) rate of convergence, which in practice allows estimation of the leverage effect on relatively short windows such as 1 month or one quarter. Estimation of the leverage effect at such short time periods is important as it allows linking the time series of the estimated leverage effect with the time series of economic and financial variables.

The two estimators we develop are complementary to each other and have their own advantages and disadvantages. The IRL is more efficient in the sense that it has a faster rate of convergence, but it requires (i) the availability of the data on a volatility instrument, and (ii) that Assumption 2 adequately describes the data at hand. In Section 4.1, we argue that this assumption holds in a reasonably broad class of models. We stress that in particular it holds for various popular parametric models used in the derivative pricing literature. We provide a simple high-frequency Durbin–Wu–Hausman test that allows one to evaluate the validity of Assumption 2, see Durbin (1954), Wu (1973), and Hausman (1978). We also discuss settings where this assumption does not hold, and illustrate the impact of that on the IRL. The advantage of the PRL is that it does not rely on the availability of the additional data or Assumption 2. The cost of this robustness is a substantially lower precision. The precision can of course be increased by using longer time intervals, and we find that the PRL becomes practical if used over time periods of multiple years of data. (The finding that the price-only estimator requires multiple years of data for precise estimation appears to be in line with earlier empirical analysis by Aït-Sahalia, Fan, and Li (2013) who use four years of 1 min data on the S&P 500 futures.) Over 11 years, the standard error for the leverage effect of the S&P 500 is 0.073.

We illustrate the finite sample performance of the estimators in Monte Carlo simulations and an empirical application. In the Monte Carlo section, we consider the estimation of the leverage effect in models where Assumption 2 holds, and in a model where this assumption fails. Higher precision of the IRL makes it possible to investigate how the leverage effect changes over time, which is important for financial applications. We provide a time series of monthly leverage effects of the S&P 500 index using the VIX as a volatility instrument. (We also use the Black–Scholes implied volatility constructed from intraday S&P 500 options in an earlier draft. The two time series of estimates share a similar pattern.) We also conduct a time series regression with the estimated leverage effect and important financial indicators, such as the credit risk, illiquidity, and the debt-to-equity ratio. Overall, we find that the leverage effect of the S&P 500 index tends to be stronger in bad times. This finding is consistent with, for example, Bandi and Renò (2012) who documented that the leverage effect of the S&P 500 is stronger when volatility is higher. Our empirical findings support the leverage hypothesis of Black (1976), while also suggesting that the debt-to-equity ratio is likely not the only determinant of the financial leverage.

For more than two decades, parametric models have been used to capture the leverage effect of daily stock returns. For example, the popular exponential generalized autoregressive conditional heteroscedasticity (EGARCH) model of Nelson (1991) is motivated by the inability of the standard GARCH models to capture the leverage effect. Many articles have also estimated the leverage effect in parametric stochastic volatility models. Such models assume a constant leverage effect, and their estimation involves either the Markov chain Monte Carlo algorithm or particle filters, see, for example, Jacquier, Polson, and Rossi (2004), Yu (2005), Pitt, Malik, and Doucet (2014), and Eraker (2004), or moment- or likelihood-based approaches, see, for example, Harvey and Shephard (1994), Pan (2002), and Aït-Sahalia and Kimmel (2007). These parametric leverage

effect estimates depend on the specified volatility dynamics. In contrast, our framework is more agnostic about the dynamics of the volatility.

The use of the derivative information to estimate the leverage effect has been previously considered by a few articles. Most closely related article is an empirical study by Andersen, Bondarenko, and Gonzalez-Perez (2015a). Although authors mainly focus on developing an alternative volatility proxy “Corridor VIX,” they also use the IRL estimator in the empirical study. Bollerslev, Sizova, and Tauchen (2012) calculated the leverage effect as a correlation between returns and changes of the VIX. Their estimates using the VIX are substantially more stable than the estimates using absolute returns as a volatility proxy in Bollerslev, Litvinova, and Tauchen (2006). Among the articles that propose fully parametric estimators, Aït-Sahalia and Kimmel (2007) also used the VIX, whereas Bakshi, Cao, and Chen (1997), Pan (2002), Eraker (2004), and Broadie, Chernov, and Johannes (2007) used S&P 500 options.

Several articles are related to the PRL estimator. Aït-Sahalia, Fan, and Li (2013) noted that empirically the correlation between returns and changes of the estimated volatility from high-frequency data is close to zero. They called this phenomenon “the leverage effect puzzle.” In the parametric framework of the Heston model, they showed that naive correlation estimator is biased. Wang and Mykland (2014), Vetter (2015), and Vetter (2012) provided estimators for the integrated covariation between the returns and their volatilities as well as for the integrated volatility of volatility. A correlation-type combination of their estimators provides a nonparametric equivalent to the bias-corrected estimator of Aït-Sahalia, Fan, and Li (2013), and converges to a certain volatility-weighted leverage effect measure. (In an earlier draft, we have provided the joint central limit theorem results for these components in a more general setting with jumps, and developed the asymptotic distribution of the estimator of this volatility-weighted leverage effect measure, VWIL (see Equation (5)).) However, empirically this estimator produces insignificant estimates of the leverage effect with 11 years of 5 min S&P 500 returns. In contrast, our PRL gives significant estimates in the same setting. This increase in precision of the PRL estimator is not surprising because it is constructed similarly to the efficient quarticity estimator in Jacod and Rosenbaum (2013), by aggregating a sequence of local estimates. To derive the asymptotic distribution of the PRL, we prove a general central limit theorem that significantly extends the results by Jacod and Rosenbaum (2013) and Aït-Sahalia and Jacod (2014). This theoretical result is of own interest. Bandi and Renò (2012), Bandi and Renò (2015), and Aït-Sahalia et al. (2015) estimated related quantities, which are however different from the integrated leverage effect.

Both PRL and IRL are related to the literature on statistical inference based on preliminary estimation of spot variances and covariances. This literature dates back to Comte and Renault (1998) and Kristensen (2010), with Jacod and Rosenbaum (2013) being a more recent contribution. The error from this preliminary estimation is asymptotically negligible if the time span diverges sufficiently fast, see, for example, Bandi and Renò (2011) and Li and Patton (2015). Our asymptotic approximation does not use a diverging time span and therefore requires taking into account the bias due to the preliminary estimation of variances and covariances. Empirically, we find that the effect of

the bias correction for the PRL is sizeable even when the time span is 11 years.

The rest of the article is organized as follows. Section 2 describes the model and the quantities of interest. Section 3 presents the PRL estimator and the associated asymptotic theory. Section 4 presents the IRL estimator and its asymptotic properties. Section 5 presents a Durbin–Wu–Hausman specification test. Section 6 provides Monte Carlo evidence and Section 7 presents empirical findings. Section 8 concludes. The online appendices contain the proofs.

## 2. Leverage Effect in Continuous Time

We work in a general nonparametric framework that allows for potential jumps in prices and volatility. This framework is commonly used in high-frequency econometrics, see, for example, Ait-Sahalia and Jacod (2014).

*Assumption 1.* Suppose that  $X$ , the logarithm of the underlying asset price, follows an Itô semimartingale,

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{E}} \delta(s, z) \mu(ds, dz), \tag{1}$$

and its spot variance, denoted by  $c_t = \sigma_t^2$ , follows:

$$c_t = c_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma}_s d\tilde{W}_s + \int_0^t \int_{\mathbb{E}} \tilde{\delta}(s, z) \mu(ds, dz), \tag{2}$$

where  $\mu(ds, dz)$  is a Poisson random measure on  $\mathbb{R}_+ \times \mathbb{E}$ .  $\mathbb{E}$  is an arbitrary Polish space with a  $\sigma$ -finite and infinite measure  $\lambda$  having no atoms, with an intensity measure  $\nu(ds, dz) = ds \otimes \lambda(dz)$ .  $W$  and  $\tilde{W}$  are standard Brownian motions. The correlation between  $W_s$  and  $\tilde{W}_s$  is  $\rho_s$ . Moreover,  $|\delta(\omega, t \wedge \tau_m(\omega), z)| \wedge 1 \leq \Gamma_m(z)$  and  $|\tilde{\delta}(\omega, t \wedge \tau_m(\omega), z)| \wedge 1 \leq \Gamma_m(z)$ , for all  $(\omega, t, z)$ , where  $(\tau_m)$  is a localizing sequence of stopping times and, for some  $r \in [0, 1]$ , the function  $\Gamma_m$  on  $\mathbb{E}$  satisfies  $\int_{\mathbb{E}} \Gamma_m(z)^r \lambda(dz) < \infty$ , that is, the jumps are summable. In addition,  $b_s$  and  $\tilde{b}_s$  are locally bounded, and  $c_s^{(c)} = \tilde{\sigma}_s^2$  is càdlàg and locally bounded. For any  $s \in [0, t]$ ,  $c_s, c_{s-}, c_s^{(c)}$ , and  $c_{s-}^{(c)}$  are almost surely positive. Also,  $[X, X]_s^c \neq 0$  and  $[c, c]_s^c \neq 0$  hold almost surely, for each  $s$ . (The superscript  $c$  denotes the continuous part of the process, and  $[X, X]_s$  is the quadratic variation of  $X$  over  $[0, s]$ .) Finally, we assume that  $\rho_s$  is càdlàg, and that  $|\rho_s|$  and  $|\rho_{s-}|$  are almost surely smaller than 1.

What is an appropriate measure of the leverage effect in this general framework? To motivate, note that in the special case of the popular Heston model, the leverage effect is usually associated with a parameter  $\rho$ , which equals

$$\rho = \lim_{\Delta \rightarrow 0} \text{Corr}(c_{s+\Delta} - c_s, X_{s+\Delta} - X_s), \quad \forall s \in [0, t]. \tag{3}$$

It also coincides with the correlation between the two Brownian motions of the Heston model, which is assumed constant over time. In general, this correlation  $\rho_s$  varies over time. We hence call it the spot leverage effect. Then, a natural measure of the

leverage effect over the interval  $[0, t]$  is a scaled integral of the spot leverage over time, namely, the integrated leverage effect,

$$\text{IL}_t = \frac{1}{t} \int_0^t \rho_s ds. \tag{4}$$

Moreover, we can represent  $\rho_s$  alternatively as

$$\rho_s \equiv \frac{[X, c]_s^{c'}}{\sqrt{[X, X]_s^{c'}} \sqrt{[c, c]_s^{c'}}},$$

where  $\prime$  denotes a derivative with respect to time.

Our IL measure is invariant to nonlinear transformations. In other words, for any smooth and monotone increasing functions  $f$  and  $g$  with nonvanishing derivatives, by Itô’s lemma, we have

$$\frac{[X, c]_s^{c'}}{\sqrt{[X, X]_s^{c'}} \sqrt{[c, c]_s^{c'}}} = \frac{[f(X), g(c)]_s^{c'}}{\sqrt{[f(X), f(X)]_s^{c'}} \sqrt{[g(c), g(c)]_s^{c'}}}.$$

As a result, we have

$$\begin{aligned} & \int_0^t \frac{[X, c]_s^{c'}}{\sqrt{[X, X]_s^{c'}} \sqrt{[c, c]_s^{c'}}} ds \\ &= \int_0^t \frac{[f(X), g(c)]_s^{c'}}{\sqrt{[f(X), f(X)]_s^{c'}} \sqrt{[g(c), g(c)]_s^{c'}}} ds. \end{aligned}$$

An important consequence is that variance-based leverage effect coincides with volatility-based leverage effect.

An alternative leverage effect measure can be defined as the scaled continuous part of the quadratic covariation between  $c$  and  $X$ ,

$$\text{VWIL}_t = \frac{[c, X]_t^c}{\sqrt{[X, X]_t^c} \sqrt{[c, c]_t^c}} = \frac{\int_0^t \rho_s \sigma_s \tilde{\sigma}_s ds}{\sqrt{\int_0^t \sigma_s^2 ds} \sqrt{\int_0^t \tilde{\sigma}_s^2 ds}}. \tag{5}$$

This leverage effect measure depends also on the path of the spot volatility and the volatility of volatility, hence we name it the volatility-weighted integrated leverage effect (VWIL). This measure is only invariant to linear transformations.

While we do allow for general price and volatility jumps, they do not contribute to our definition of leverage. We choose not to include them for the following reasons. First, the IL is based on the spot correlation, which is an intuitive generalization of the  $\rho$  parameter in the Heston model, and which endows the IL measure with invariance to smooth transformations. However, the spot correlation is not well defined without the exclusion of jumps. (Only the continuous part of the quadratic variation is absolutely continuous and differentiable almost everywhere.) Second, Andersen, Bondarenko, and Gonzalez-Perez (2015a) pointed out that the VIX contains artificial jumps due to its implementation by the CBOE. The IRL using the VIX is robust to the artificial jumps, because it only depends on the continuous part. Third, we find in our empirical study that truncating off the jump component makes the estimates more stable. Finally, Bollerslev et al. (2009) found that the leverage effect works primarily through the continuous components. We point out that the use of our method does not require taking a stand on the importance of jumps, and can be viewed as the estimation of the continuous part of the total leverage effect. (This interpretation is analogous to the estimation of the continuous part of beta in, for example, Reiß, Todorov, and Tauchen (2015) and

Ait-Sahalia, Kalnina, and Xiu (2014), and the principal component analysis of the continuous spot covariance matrix in Ait-Sahalia and Xiu (2014).

### 3. Estimation using Price Observations Alone

The current section introduces the Price-only Realized Leverage (PRL), a nonparametric leverage effect estimator that uses price observations alone, and presents its asymptotic distribution. The PRL estimator uses preliminary estimation of volatility, and it is based on an aggregation of spot correlations between the returns and the estimated volatility increments.

To obtain preliminary estimates of spot volatility, we use an additional set of smaller and overlapping blocks, each containing  $l_n$  observations, within each larger nonoverlapping block of size  $k_n$ . We estimate the unobservable spot variance with the truncated realized variance over each smaller block,

$$\hat{\sigma}_i^2 = \frac{1}{l_n \Delta_n} \sum_{j=0}^{l_n-1} \left( \Delta_{i+j}^n X \right)^2 \mathbf{1}_{\left\{ \left| \Delta_{i+j}^n X \right| \leq u_n \right\}},$$

with the threshold  $u_n$  satisfying conditions of Theorem 1. The use of overlapping blocks of size  $l_n$  is motivated by the fact that the nonoverlapping alternative would be less efficient in the sense of having a larger asymptotic variance, as shown by Ait-Sahalia and Jacod (2014) for the estimation of the VWIL. Meanwhile, the use of overlapping blocks of size  $k_n$  would not improve the efficiency of the PRL, similar in spirit to the estimator of Jacod and Rosenbaum (2013).

To simplify the exposition, we denote

$$C_s = \begin{pmatrix} c_s & c_s^{(X,c)} \\ c_s^{(X,c)} & c_s^{(c)} \end{pmatrix} \equiv \begin{pmatrix} [X, X]_s^{c'} & [X, \sigma^2]_s^{c'} \\ [X, \sigma^2]_s^{c'} & [\sigma^2, \sigma^2]_s^{c'} \end{pmatrix}, \quad (6)$$

and notice that it implies  $c_s^{(X,c)} = \rho_s c_s^{1/2} (c_s^{(c)})^{1/2}$ . We can approximate each element of this matrix at  $s = ik_n \Delta_n$  as follows (we use  $\hat{c}$  to denote the estimator of  $\sigma^2$  based on a block of size  $k_n$ , and use  $\hat{\sigma}^2$  when a block of size  $l_n$  is used):

$$\begin{aligned} \hat{c}_{ik_n+1} &= \frac{1}{k_n \Delta_n} \sum_{j=0}^{k_n-1} \left( \Delta_{ik_n+1+j}^n X \right)^2 \mathbf{1}_{\left\{ \left| \Delta_{ik_n+1+j}^n X \right| \leq u_n \right\}}, \\ \hat{c}_{ik_n+1}^{(X,c)} &= \frac{1}{l_n k_n \Delta_n} \sum_{j=0}^{k_n-1} \left\{ \left( \hat{\sigma}_{ik_n+1+j+l_n}^2 - \hat{\sigma}_{ik_n+1+j}^2 \right) \right. \\ &\quad \left. \times \sum_{l=0}^{2l_n-1} \left( \Delta_{ik_n+1+j+l}^n X \right) \mathbf{1}_{\left\{ \left| \Delta_{ik_n+1+j+l}^n X \right| \leq u_n \right\}} \right\}, \\ \hat{c}_{ik_n+1}^{(c)} &= \frac{3}{2l_n k_n \Delta_n} \sum_{j=0}^{k_n-1} \left\{ \left( \hat{\sigma}_{ik_n+1+j+l_n}^2 - \hat{\sigma}_{ik_n+1+j}^2 \right)^2 \right. \\ &\quad \left. - \frac{4}{l_n} \left( \hat{\sigma}_{ik_n+1+j}^2 \right)^2 \right\}, \end{aligned}$$

therefore the estimator of the spot leverage effect is

$$\hat{\rho}_{ik_n+1} = \frac{\hat{c}_{ik_n+1}^{(X,c)}}{\left( \hat{c}_{ik_n+1} \right)^{1/2} \left( \hat{c}_{ik_n+1}^{(c)} \right)^{1/2}}.$$

Our PRL can be constructed as follows:

$$\begin{aligned} \text{PRL}_t &= \frac{k_n \Delta_n}{t} \sum_{i=0}^{N_t^n} \left( \hat{\rho}_{ik_n+1} - \frac{1}{2k_n \sqrt{\Delta_n}} \left( -\frac{5}{\beta} \frac{\hat{c}_{ik_n+1}^{3/2} \hat{c}_{ik_n+1}^{(X,c)}}{\left( \hat{c}_{ik_n+1}^{(c)} \right)^{3/2}} \right. \right. \\ &\quad \left. \left. - \frac{151\beta}{120} \frac{\hat{c}_{ik_n+1}^{(X,c)}}{\hat{c}_{ik_n+1}^{1/2} \left( \hat{c}_{ik_n+1}^{(c)} \right)^{1/2}} + \frac{36}{\beta} \frac{\hat{c}_{ik_n+1}^{(X,c)} \hat{c}_{ik_n+1}^{7/2}}{\left( \hat{c}_{ik_n+1}^{(c)} \right)^{3/2}} \right. \right. \\ &\quad \left. \left. + \frac{9}{\beta} \frac{\hat{c}_{ik_n+1}^{(X,c)} \hat{c}_{ik_n+1}^{3/2}}{\left( \hat{c}_{ik_n+1}^{(c)} \right)^{1/2}} + \frac{453\beta}{280} \frac{\hat{c}_{ik_n+1}^{(X,c)} \left( \hat{c}_{ik_n+1}^{(c)} \right)^{1/2}}{\hat{c}_{ik_n+1}^{1/2}} \right) \right), \quad (7) \end{aligned}$$

where  $N_t^n = \lfloor (t/\Delta_n - 2l_n + 2)/k_n \rfloor - 1$  and  $\beta = l_n \sqrt{\Delta_n}$ . The component after  $\hat{\rho}_{ik_n+1}$  corrects an asymptotic bias. The next theorem presents the asymptotic distribution of the PRL estimator.

*Theorem 1.* Suppose Assumption 1 holds. In addition,  $\sigma_t$  is a continuous Itô semimartingale. Suppose  $\beta = l_n \sqrt{\Delta_n} \in (0, \infty)$ ,  $k_n^3 \Delta_n^2 \rightarrow \infty$ , and  $k_n^4 \Delta_n^3 \rightarrow 0$ . Moreover, suppose  $u_n \asymp \Delta_n^r$  with  $0 \leq r < 1/3$  and  $5/(12 - 6r) \leq \varpi < 1/2$ . Then,

$$\Delta_n^{-1/4} (\text{PRL}_t - \text{IL}_t) \xrightarrow{L-s} \mathcal{Z} \sqrt{V_t^{\text{PRL}}},$$

where  $\mathcal{Z}$  is a standard normal random variable defined on the extension of the original probability space, and the variance  $V_t^{\text{PRL}}$  is given by

$$\begin{aligned} V_t^{\text{PRL}} &= \frac{1}{t^2} \int_0^t \left( \frac{23\beta}{15} + \frac{8}{3\beta} \frac{c_s^2}{c_s^{(c)}} - \frac{2}{\beta} \frac{c_s (c_s^{(X,c)})^2}{(c_s^{(c)})^2} \right. \\ &\quad \left. + \frac{12}{\beta^3} \frac{c_s^3 (c_s^{(X,c)})^2}{(c_s^{(c)})^3} + \frac{\beta}{21} \frac{(c_s^{(X,c)})^2}{c_s c_s^{(c)}} \right) ds. \end{aligned}$$

Similarly, we can construct a consistent estimator of  $V_t^{\text{PRL}}$  using local estimators of  $c_{ik_n+1}$ ,  $c_{ik_n+1}^{(X,c)}$ , and  $c_{ik_n+1}^{(c)}$  as follows:

$$\begin{aligned} \hat{V}_t^{\text{PRL}} &= \frac{k_n \Delta_n}{t^2} \sum_{i=0}^{N_t^n} \left( \frac{23\beta}{15} + \frac{8}{3\beta} \frac{\hat{c}_{ik_n+1}^2}{\hat{c}_{ik_n+1}^{(c)}} - \frac{2}{\beta} \frac{\hat{c}_{ik_n+1} \left( \hat{c}_{ik_n+1}^{(X,c)} \right)^2}{\left( \hat{c}_{ik_n+1}^{(c)} \right)^2} \right. \\ &\quad \left. + \frac{12}{\beta^3} \frac{\hat{c}_{ik_n+1}^3 \left( \hat{c}_{ik_n+1}^{(X,c)} \right)^2}{\left( \hat{c}_{ik_n+1}^{(c)} \right)^3} + \frac{\beta}{21} \frac{\left( \hat{c}_{ik_n+1}^{(X,c)} \right)^2}{\hat{c}_{ik_n+1} \hat{c}_{ik_n+1}^{(c)}} \right). \quad (8) \end{aligned}$$

Appendix A contains the proof of Theorem 1. Note that due to the preliminary estimation of volatility, the rate of convergence of the PRL estimator is slower than the usual  $\sqrt{n}$  rate of convergence.

To prove Theorem 1, we first prove a more general result about estimating  $\int_0^t g(C_s) ds$  for any smooth function  $g(\cdot)$ ; see Theorem A1 in Appendix A. This theorem extends Jacod and Rosenbaum (2013) who estimated  $\int_0^t g(c_s) ds$ . The spot matrix  $C_s$  in (6) not only contains the spot volatility  $c_s$  of the observable process  $X$ , but also volatilities of unobservable processes. Therefore, estimating integrals of functionals of  $C_s$  requires substantially more involved methods and first-order bias corrections. Theorem 1 is related to Wang and Mykland (2014), Vetter (2015), and chap. 8 of Ait-Sahalia and Jacod (2014), in that our

$\hat{c}_{ik_n+1}^{(X,c)}$  and  $\hat{c}_{ik_n+1}^{(c)}$ , for each block  $[(ik_n + 1)\Delta_n, (i + 1)k_n\Delta_n]$ , are similar to their estimators on  $[0, t]$ .

The above theorem assumes that there are no volatility jumps. Volatility jumps do not affect the estimation of the covariation between the continuous components of returns and volatilities,  $c_{ik_n+1}^{(X,c)}$ , because these jumps do not co-vary with the continuous component of returns, see, for example, Ait-Sahalia et al. (2015). However, volatility jumps affect the estimation of  $\hat{c}_{ik_n+1}^{(c)}$ , because its estimator would converge to the total quadratic variation of the volatility process. To ensure a fair comparison between the IRL and PRL estimates in the empirical work, we suggest identifying volatility jumps using the jumps of the VIX, a strategy previously adopted by Todorov and Tauchen (2011) for the estimation of the volatility jump activity index. We thus modify the above  $\hat{c}_{ik_n+1}^{(c)}$  as follows:

$$\tilde{c}_{ik_n+1}^{(c)} = \frac{3}{2l_n k_n \Delta_n} \sum_{j=0}^{k_n-1} \left\{ (\hat{\sigma}_{ik_n+1+j+l_n}^2 - \hat{\sigma}_{ik_n+1+j}^2)^2 - \frac{4}{l_n} (\hat{\sigma}_{ik_n+1+j}^2)^2 \right\} 1_{\{|Z_{ik_n+1+j+l_n} - Z_{ik_n+1+j}| \leq v_n\}},$$

where  $Z$  is the VIX and  $v_n \asymp (l_n \Delta_n)^\omega$ . In the empirical application, we find  $\tilde{c}^{(c)}$  and  $\hat{c}^{(c)}$  to be very close. (Unlike price jumps, volatility jumps are much more complicated to deal with. Proving the asymptotic properties of this estimator may be possible by combining our results with a method used by Jacod and Rosenbaum (2015), which would complicate the proof even further.) We also recommend a small-sample adjustment, which is to divide the above sum by the number of nonzero summands instead of  $k_n$ . It is not needed given the sample size in our empirical study.

#### 4. Estimation with a Volatility Instrument

##### 4.1. Volatility Instrument

Financial derivatives are known to contain valuable information about the volatility. However, to make use of this information, certain assumptions that link the volatility dynamics under the risk-neutral measure  $\mathbb{Q}$  to the dynamics under the objective measure  $\mathbb{P}$  are necessary. In this section, we assume that we have access to what we call a volatility instrument.

*Assumption 2.* There exists an observable variable  $Z_t$ , which is a monotone increasing and twice differentiable function of  $c_t$  and a smooth function of  $t$ , that is,  $Z_t = f(t, c_t)$ . In addition,  $[Z, Z]_s^c$  is almost surely not vanishing and  $f'(s, x) > 0$ , for all  $0 \leq s \leq t$  and  $x > 0$ . (We use  $f'$  here to denote the derivative of  $f$  with respect to the second argument.) We call such a variable a volatility instrument.

Assumption 2 implies that the spot leverage effect between  $X_t$  and  $c_t$  equals the spot leverage effect between  $X_t$  and  $Z_t$ ,

$$\rho_s = \frac{[X, c]_s^{c'}}{\sqrt{[X, X]_s^{c'}} \sqrt{[c, c]_s^{c'}}} = \frac{[X, Z]_s^{c'}}{\sqrt{[X, X]_s^{c'}} \sqrt{[Z, Z]_s^{c'}}}.$$

The last equality is true by Itô's lemma for general Itô semimartingales, see, for example, Proposition 8.19 of Cont and Tankov (2004). This implies that the IL measure is invariant

with respect to the functional transformation  $f$ . If the researcher has access to high-frequency data on a volatility instrument, this property can be used to eliminate the effect of risk premia embedded in the function  $f$ , and hence to estimate IL.

*Remark.* A weaker assumption is sufficient for all of the results in this article. If we decompose the stochastic process  $Z_t$  into its continuous and jump components, we only require  $Z_t^c = \tilde{f}(t, c_t)$ , where  $Z_t^c$  is the continuous component of  $Z_t$ . This means we need no assumptions on the jump part of the volatility instrument other than some mild conditions on its activity as in Assumption 1. For ease of presentation and interpretation, we adopt the current Assumption 2.

Assumption 2 imposes restrictions on the volatility dynamics under the risk-neutral measure  $\mathbb{Q}$  and is therefore a high-level assumption. It holds for a number of models in the derivative pricing literature. We summarize two classes of such risk-neutral models for the spot variance, of which the implied variance,  $Z_t$ , defined as the squared VIX up to some scalar, is a monotone increasing and differentiable function. (For both classes of models, the dynamics of the logarithm of the index  $X_t$ , under the risk-neutral measure  $\mathbb{Q}$ , is assumed to follow

$$X_t = X_0 + \int_0^t b_s^{\mathbb{Q}} ds + \int_0^t \sigma_s dW_s^{\mathbb{Q}} + \int_0^t \int_{\mathbb{R}} z (N(ds, dz) - \nu(V_s, dz) ds),$$

where the drift  $b_s^{\mathbb{Q}}$  is determined by the no-arbitrage condition and is irrelevant for the pricing of the VIX,  $W^{\mathbb{Q}}$  is a  $\mathbb{Q}$ -Brownian motion, and  $N(ds, dz)$  is the jump measure of  $X_t$  with compensator  $\nu(V_s, dz) ds$  that may depend on the spot variance. We assume there exist constants  $\eta_0$  and  $\eta_1$ , such that for all  $v > 0$ ,  $\int_{\mathbb{R}} z^2 \nu(v, dz) = \eta_0 + \eta_1 v$ .) The first class of models, henceforth *Type I models*, has the following risk-neutral dynamics:

$$\text{Type I model: } \sigma_t^2 = \sigma_0^2 + \int_0^t \kappa (\bar{\xi}^2 - \sigma_s^2) ds + M_t^{\mathbb{Q}},$$

where  $M^{\mathbb{Q}}$  is a martingale under the  $\mathbb{Q}$ -measure and  $\kappa, \bar{\xi}$  are model parameters. Type I models include those studied by Bakshi, Cao, and Chen (1997), Bates (2000), Pan (2002), Eraker (2004), Eraker, Johannes, and Polson (2003), Broadie, Chernov, and Johannes (2007), and Bates (2012), among others, where jumps may be driven by the compound Poisson process with time-varying intensity or the CGMY process (Carr et al. 2003). Type I models also include non-Gaussian Ornstein–Uhlenbeck (OU) processes considered by Barndorff-Nielsen and Shephard (2001); see also Shephard (2005) for a collection of similar models. For Type I models, we can show that  $Z_t = a + b\sigma_t^2$ , where  $a$  and  $b$  depend on parameters that appear in the risk-neutral dynamics.

Second, *Type II models* impose an exponential-affine structure for the  $\mathbb{Q}$ -dynamics of  $\sigma_t^2$ :

$$\text{Type II model: } \log \sigma_t^2 = \alpha + \beta F_t, \quad F_t = F_0 - \int_0^t \kappa F_s ds + L_t^{\mathbb{Q}},$$

where  $L_t^{\mathbb{Q}}$  is a finite variational Lévy martingale with diffusive coefficient  $\sigma$  and Lévy measure  $\tilde{\nu}$ , and  $\alpha, \beta, \kappa$  are model parameters. This model dates back to Nelson (1990), who introduces it as a continuous-time limit of the discrete EGARCH model. Andersen, Bollerslev, and Meddahi (2005) and Chernov et al. (2003) had employed this model in their empirical work. One can show that Type II models imply the following pricing formula for the implied variance,

$$Z_t = \eta_0 + \frac{1}{\tau} \int_0^\tau (\eta_1 + 1) \times \exp(\alpha + e^{-\kappa u}(\log \sigma_t^2 - \alpha) + C(u)) du,$$

where  $\tau = 21$  trading days is the horizon of the VIX,  $C(u) = \int_0^u \varphi(\beta e^{-\kappa v}) dv$ ,  $\varphi(u) = \sigma^2 u^2 / 2 + \int_{\mathbb{R}} (e^{uz} - 1 - uz) \tilde{\nu}(dz)$ , and  $\eta_0$  and  $\eta_1$  are constants defined in the remark for Assumption 2. It can be easily verified that  $Z_t$  is an increasing differentiable function of  $\sigma_t^2$ . In fact,  $Z_t \approx a + b\sigma_t^{2d}$ , where  $a, b$ , and  $d$  are some constants that depend on the parameters in the risk-neutral dynamics.

Assumption 2 effectively says that to use the VIX as an instrument, the VIX should only be driven by the unobserved volatility. Notice that this does not necessarily imply that the objective volatility process can only be driven by one factor. However, it does rule out a few models in the recent empirical finance literature, see, for example, Mencia and Sentana (2013), Christoffersen, Heston, and Jacobs (2009), and Andersen, Fusari, and Todorov (2015b). For example, a common two-factor volatility process under the risk-neutral measure  $\mathbb{Q}$  is

$$\begin{aligned} d\sigma_t^2 &= (\eta + \kappa \xi_t^2 - \kappa^{\mathbb{Q}} \sigma_t^2) dt + \gamma \sigma_t d\tilde{W}_t^{\mathbb{Q}}, \\ d\xi_t^2 &= (\kappa_{\xi} \bar{\xi}^2 - \kappa_{\xi}^{\mathbb{Q}} \xi_t^2) dt + \gamma_{\xi} \xi_t d\tilde{B}_t^{\mathbb{Q}}, \end{aligned} \tag{9}$$

$$\hat{\rho}_{ik_n+1} = \frac{\sum_{j=0}^{k_n-1} (\Delta_{ik_n+1+j}^n Z) (\Delta_{ik_n+1+j}^n X) \cdot \mathbf{1}_{\{|\Delta_{ik_n+1+j}^n Z| \leq u_n\}} \mathbf{1}_{\{|\Delta_{ik_n+1+j}^n X| \leq u_n\}}}{\sqrt{\sum_{j=0}^{k_n-1} (\Delta_{ik_n+1+j}^n Z)^2 \mathbf{1}_{\{|\Delta_{ik_n+1+j}^n Z| \leq u_n\}}} \sqrt{\sum_{j=0}^{k_n-1} (\Delta_{ik_n+1+j}^n X)^2 \mathbf{1}_{\{|\Delta_{ik_n+1+j}^n X| \leq u_n\}}}, \tag{10}$$

where  $E^{\mathbb{Q}}(d\tilde{B}_t^{\mathbb{Q}} d\tilde{W}_t^{\mathbb{Q}}) = \tilde{\rho}_t dt$ , and  $E^{\mathbb{Q}}(d\tilde{B}_t^{\mathbb{Q}} dW_t^{\mathbb{Q}}) = \bar{\rho}_t dt$ . It is straightforward to derive that  $VIX_t^2 = a + b\sigma_t^2 + c\xi_t^2$ , where  $a, b$ , and  $c$  depend on parameters under the risk-neutral measure  $\mathbb{Q}$ . The VIX-based IRL estimator converges to

$$\frac{1}{t} \int_0^t \frac{b\gamma \rho_s \sigma_s + c\gamma_{\xi} \bar{\rho}_s \xi_s}{\sqrt{b^2 \gamma^2 \sigma_s^2 + c^2 \gamma_{\xi}^2 \xi_s^2}} ds \neq \frac{1}{t} \int_0^t \rho_s ds = \Pi_t.$$

The typical justification for one of the two factors, say  $\xi_t^2$ , is that it captures the dynamics of the long-term variance, which has a much smaller  $\gamma_{\xi}$  compared to  $\gamma$ , as implied from the empirical data, see, for example, Song and Xiu (2014). In addition,  $c$  is smaller relative to  $b$ , since the maturity of the VIX is only one month. As a result, the IRL estimator may approximate IL very well, despite being inconsistent (this is also what we find in our simulations).

Besides the VIX, alternative volatility instruments can also be used. For example, for the S&P 500 index, we can use the intraday VIX futures or VIX options, which require a stronger

assumption that the state-price densities of the implied variance only depend on  $\sigma_t^2$ . For individual equities, the VIX can be calculated. (The CBOE has already calculated the VIX for a limited number of stocks since January 7, 2011.) Alternatively, the Black–Scholes implied volatility with a fixed maturity and moneyness can be used as a volatility instrument. Assumption 2 is satisfied if the marginal state price density of  $X_t$  only depends on  $\sigma_t^2$ , and the option price is homogenous of degree 1, see, for example, Joshi (2002) and Song and Xiu (2014). More specifically, suppose that

$$C(X, \sigma_t^2, k, T - t) = \exp(X) \cdot C(0, \sigma_t^2, m, T - t),$$

where  $C$  is the price of a European call option with a log strike price  $k$  and a fixed maturity date  $T$ , and  $m = k - X$  is a fixed log-moneyness. (Since  $X$  is time-varying, fixed log-moneyness implies that  $k$  needs to be changed over time.) It then follows that the Black–Scholes implied volatility is a deterministic function of  $\sigma_t^2$ .

#### 4.2. IRL Estimator and Its Asymptotic Distribution

We now develop the asymptotic theory for the IRL, which exploits additional information embedded in the instrument. Our asymptotic results of the IRL estimator draw extensively from the general theoretical framework of Jacod and Rosenbaum (2013).

Suppose we have a sample of equidistant observations on  $X$  and  $Z$  over the interval  $[0, t]$ . Denote the distance between adjacent observations by  $\Delta_n$ . Partition all observations into nonoverlapping blocks, indexed by  $i = 0, 1, 2, \dots, [t/(\Delta_n k_n)]$ , so that each block contains  $k_n$  observations. We can then estimate the spot leverage effect at time  $ik_n \Delta_n$  using the local truncated realized correlation between  $X$  and  $Z$ ,

where we truncate jumps in  $X$  and  $Z$  using thresholds  $u_n$  and  $u'_n$ . The choice of  $u_n$  and  $u'_n$  is standard in the literature, see, for example, Aït-Sahalia and Jacod (2014). It is crucial to truncate jumps here as the true spot correlation is only defined for the continuous components. As the length of the local window  $k_n \Delta_n$  shrinks to zero,  $\hat{\rho}_{ik_n+1}$  approximates the spot leverage  $\rho_{ik_n \Delta_n}$ ,

$$\begin{aligned} \hat{\rho}_{ik_n+1} &\approx \frac{[Z, X]_s^{c'}}{\sqrt{[Z, Z]_s^{c'}} \sqrt{[X, X]_s^{c'}}} \\ &= \frac{f'(s, \sigma_s^2) [X, \sigma^2]_s^{c'}}{\sqrt{[f'(s, \sigma_s^2)]^2 [\sigma^2, \sigma^2]_s^{c'}} \sqrt{[X, X]_s^{c'}}} \\ &= \frac{[X, \sigma^2]_s^{c'}}{\sqrt{[\sigma^2, \sigma^2]_s^{c'}} \sqrt{[X, X]_s^{c'}}} = \rho_s, \text{ where } s = ik_n \Delta_n. \end{aligned} \tag{11}$$

Notice how this relationship holds for any smooth function  $f$ . In particular, the spot leverage  $\rho_s$  is invariant to functional transformations of  $\sigma^2$  and  $X$ , a property that is analogous to the linear

invariant property of the standard correlation of two random variables. This property is important in that it eliminates the impact of risk premia embedded in the pricing function  $f$ . A consistent estimator of the IL is thereby a Riemann sum of the estimators of the spot leverage, but it has an asymptotic bias.

Define the IRL as follows:

$$\text{IRL}_t = \frac{k_n \Delta_n}{t} \sum_{i=0}^{\lfloor t/k_n \Delta_n \rfloor - 1} \left( \hat{\rho}_{ik_n+1} - \frac{1}{2k_n} \left( (\hat{\rho}_{ik_n+1})^3 - \hat{\rho}_{ik_n+1} \right) \right). \quad (12)$$

The following theorem presents the asymptotic distribution of the IRL estimator.

*Theorem 2.* Suppose Assumptions 1 and 2 hold. Let  $u_n \asymp \Delta_n^\varpi$ ,  $u'_n \asymp \Delta_n^\varpi$ , and suppose  $5/(12 - 2r) \leq \varpi < 1/2$ . Then, for any fixed  $t$ , and as  $\Delta_n \rightarrow 0$ ,  $k_n^2 \Delta_n \rightarrow 0$  and  $k_n^3 \Delta_n \rightarrow \infty$ , we have

$$\Delta_n^{-1/2} (\text{IRL}_t - \text{IL}_t) \xrightarrow{L-s} \mathcal{Z} \sqrt{V_t^{\text{IRL}}},$$

where 
$$V_t^{\text{IRL}} = \frac{1}{t^2} \int_0^t (1 - 2\rho_s^2 + \rho_s^4) ds,$$

and  $\mathcal{Z}$  is a standard normal random variable defined on the extension of the original probability space. (The convergence here is stable in law, see, for example, chap. 2.2.1 of Jacod and Protter (2012) for a detailed review.)

The estimator of the asymptotic variance can be constructed similarly,

$$\hat{V}_t^{\text{IRL}} = \frac{k_n \Delta_n}{t^2} \sum_{i=0}^{\lfloor t/k_n \Delta_n \rfloor - 1} \left( 1 - 2(\hat{\rho}_{ik_n+1})^2 + (\hat{\rho}_{ik_n+1})^4 \right).$$

An alternative way to estimate the asymptotic variance is by applying the subsampling method of Kalnina (2011) and Kalnina (2015). While the sampling period above is fixed at  $[0, t]$  such as 1 month, we typically apply the IRL estimator to a sequence of time periods in practice. Therefore, we provide uniform confidence bands of IL across different periods. Suppose the  $\tau$ th sampling month is  $[(\tau - 1)t, \tau t]$ , and there are  $N$  months in total. Let  $\text{IL}(\tau)$  denote the integrated leverage at month  $\tau$ , and  $\text{IRL}(\tau)$  and  $V^{\text{IRL}}(\tau)$  be its estimator and the asymptotic variance. Then a nominal level  $1 - \alpha$  uniform confidence band for  $\text{IL}(\tau)$  is given by

$$\text{CI}_n(\tau; 1 - \alpha) = \left[ \text{IRL}(\tau) - z_{N,\alpha} \Delta^{1/2} \sqrt{\hat{V}^{\text{IRL}}(\tau)}, \text{IRL}(\tau) + z_{N,\alpha} \Delta^{1/2} \sqrt{\hat{V}^{\text{IRL}}(\tau)} \right],$$

where  $z_{N,\alpha}$  is the  $1 - \alpha$  quantile of the variable  $\max_{1 \leq \tau \leq N} |\mathcal{N}_\tau|$  and  $\{\mathcal{N}_\tau : 1 \leq \tau \leq N\}$  are independent standard normal variables. The asymptotic property of the above confidence band follows from the asymptotic independence of the estimation errors across nonoverlapping periods, as is typical in the high-frequency setting.

For the purpose of estimating the spot correlation, only the rate  $\Delta_n^{-1/2}$  of the tuning parameter  $k_n$  is compatible with the optimal rate of convergence, see Alvarez et al. (2012). Interestingly, Theorem 2 and its proof show that for the purpose of estimating the integrated correlation, a slower rate for  $k_n$  ensures

the feasibility of the inference by diminishing asymptotic biases, while maintaining the same asymptotic variance as in the case when  $k_n$  is of order  $\Delta_n^{-1/2}$ . The final estimator is easy to construct and has a simple expression for the asymptotic variance. Our Monte Carlo simulations suggest it also has good finite sample performance.

An alternative estimator can be constructed by using overlapping blocks of size  $k_n$ . Such an estimator shares the same asymptotic distribution with the nonoverlapping estimator. This feature has been shown by Jacod and Rosenbaum (2013) for their estimator. We use the nonoverlapping implementation because it is faster. We find that both implementations have similar finite sample performance.

## 5. Specification Test

In the presence of a valid volatility instrument, the IRL is more efficient than the PRL, in the sense that it has a faster rate of convergence. This increase in precision does not come without a cost. The IRL estimator requires the availability of a volatility instrument, which might not be available for all stocks. It also assumes that Assumption 2 adequately describes the data used. Besides, the IRL has two practical disadvantages. First, the use of two high-frequency time series creates biases due to asynchronicity. Second, the derivatives are typically less liquid than the underlying stocks, which results in higher levels of the market microstructure noise. Both issues can be mitigated by using lower sampling frequencies, which leads to data loss.

It is of interest to test whether the two estimators, the PRL and the IRL, converge to the same limit. We use a Durbin–Wu–Hausman statistic, that is, a standardized difference between the PRL and the IRL estimators. It can be used as a specification test of Assumption 2.

*Corollary 3.* Suppose Assumptions 1 and 2 hold. In addition,  $\sigma_t$  is a continuous Itô semimartingale. Let  $u_n \asymp \Delta_n^\varpi$ ,  $u'_n \asymp \Delta_n^\varpi$ , and suppose  $0 \leq r < 1/3$  and  $5/(12 - 6r) \leq \varpi < 1/2$ . Let PRL be defined by Equation (7) with  $k'_n$  observations in each of the long blocks and  $l_n$  observations in each of the short blocks. Let IRL be defined by Equation (12) with  $k_n$  observations in each block. Suppose  $\beta = l_n \sqrt{\Delta_n} \in (0, \infty)$ . Then, for any fixed  $t$ , and as  $\Delta_n \rightarrow 0$ ,  $k_n^2 \Delta_n \rightarrow 0$ ,  $k_n^3 \Delta_n \rightarrow \infty$ ,  $(k'_n)^3 \Delta_n^2 \rightarrow \infty$ , and  $(k'_n)^4 \Delta_n^3 \rightarrow 0$ , we have

$$\Delta_n^{-1/4} \frac{\text{PRL}_t - \text{IRL}_t}{\sqrt{\hat{V}_t^{\text{PRL}}}} \xrightarrow{L-s} \mathcal{Z}, \quad (13)$$

where  $\hat{V}_t^{\text{PRL}}$  is given in Equation (8) and  $\mathcal{Z}$  is a standard normal random variable defined on the extension of the original probability space.

This test is consistent against those alternatives, under which our asymptotic theory for the PRL estimator continues to hold, while the IRL estimator becomes inconsistent. Corollary 3 follows immediately from Theorems 1 and 2. The first-order asymptotic distribution of the test statistic does not reflect the estimation error in the IRL because it has a faster rate of convergence than that of the PRL. One alternative that can capture this

additional estimation error is to use a subsampling-based estimation of the asymptotic variance of the test instead of  $\hat{V}_t^{\text{PRL}}$ , see Kalnina (2015).

### 6. Simulations

This section considers the finite sample properties of the PRL, and of the IRL using the VIX as a volatility instrument.

We first consider two different models of price and volatility dynamics, the generalized Heston and the log volatility (LogV) models. In the Heston model, the log-price  $X$  under the objective measure satisfies

$$\begin{aligned} dX_t &= (\mu_0 + \mu_1\sigma_t^2)dt + \sigma_t dW_t + J_X dN_t - \lambda_t \mu_X dt, \\ d\rho_t^2 &= \kappa_\rho(\bar{\rho} - \rho_t)dt + \gamma_\rho \sqrt{1 - \rho_t^2} dB_t, \\ d\sigma_t^2 &= \kappa(\xi - \sigma_t^2)dt + \gamma\sigma_t d\tilde{W}_t + J_{\sigma^2} dN_t - \beta_{\sigma^2} \lambda_t dt, \end{aligned}$$

where  $W_t$  and  $\tilde{W}_t$  are two Brownian motions with correlation  $\rho_t$ ,  $B_t$  is another independent Brownian Motion,  $N_t$  is a Poisson process with state-dependent intensity  $\lambda_t = \lambda_0 + \lambda_1\sigma_t^2$ ,  $J_X$  is a random jump size of  $X$  satisfying  $J_X \sim N(\mu_X, \sigma_X^2)$ , and  $J_{\sigma^2}$  is a random jump size of  $\sigma_t^2$  satisfying  $J_{\sigma^2} \sim \exp(\beta_{\sigma^2})$ . We set the parameters to the values typically used in the literature:  $\kappa = 5$ ,  $\gamma = 0.35$ ,  $\kappa_\rho = 4$ ,  $\gamma_\rho = 0.2$ ,  $\bar{\rho} = -0.8$ ,  $\xi = 0.06$ ,  $\mu_X = 0$ ,  $\sigma_X = 0.05$ ,  $\beta_{\sigma^2} = 0.01$ ,  $\lambda_0 = 30$ ,  $\lambda_1 = 60$ ,  $\mu_0 = 0.05$ , and  $\mu_1 = 0.5$ . We assume the risk-neutral dynamics follows the same model, but with different parameters. In this case,  $\text{VIX}_t^2 = a + b\sigma_t^2$ , where the constants  $a$  and  $b$  depend on the parameters of the risk-neutral dynamics of  $X$ . Since our analysis depends on the risk-neutral parameters through the use of the VIX, we only need to choose the values of  $a$  and  $b$ . According to our calibration, we set  $\text{VIX}_t^2 = 100^2 \cdot (0.06 + 0.63\sigma_t^2)$ .

In the LogV model, the log-price  $X$  under the objective measure satisfies

$$\begin{aligned} dX_t &= (\mu_0 + \mu_1\sigma_t^2)dt + \sigma_t dW_t + J_X dN_t - \lambda \mu_X dt, \\ d\rho_t^2 &= \kappa_\rho(\bar{\rho} - \rho_t)dt + \gamma_\rho \sqrt{1 - \rho_t^2} dB_t, \\ dF_t &= \kappa F_t dt + \sigma d\tilde{W}_t + J_F dN_t - \mu_F \lambda dt, \\ \sigma_t^2 &= \exp(\alpha + \beta F_t), \end{aligned}$$

where again  $W_t$  and  $\tilde{W}_t$  are two Brownian motions with correlation  $\rho_t$ ,  $B_t$  is another independent Brownian Motion,  $N_t$  is a Poisson process with intensity  $\lambda$ ,  $J_X \sim N(\mu_X, \sigma_X^2)$ , and  $J_F \sim N(\mu_F, \sigma_F^2)$ . We set the values of the parameters to  $\alpha = -2.8$ ,  $\beta = 3$ ,  $\bar{\rho} = -0.8$ ,  $\mu_X = 0$ ,  $\sigma_X = 0.05$ ,  $\mu_F = 0.02$ ,  $\sigma_F = 0.02$ ,  $\sigma = 0.8$ ,  $\kappa = -4$ ,  $\lambda = 20$ ,  $\mu_0 = 0.05$ , and  $\mu_1 = 0.5$ . If the risk-neutral dynamics of  $X$  follows the same model, then  $\text{VIX}_t^2 = a + b\sigma_t^{2d}$  for some constants  $a$ ,  $b$ , and  $d$ , where  $a$ ,  $b$ , and  $d$  depends on the risk-neutral parameters. We set  $\text{VIX}_t^2 = 100^2 \cdot (0.1 + 0.75\sigma_t^{2 \times 0.8})$ .

We fix the truncation levels (values of  $u_n$  and  $u'_n$ ) to be  $3\sqrt{[Y, Y]_t^c / t \Delta_n^{0.47}}$ , where  $Y$  is either the return of the stock or the change in the VIX, and  $[Y, Y]_t^c$  is estimated by bipower variation, see Barndorff-Nielsen and Shephard (2004) and Ait-Sahalia and Jacod (2012). For the IRL estimator, the rule-of-thumb choice of the block size  $k_n$  is approximately between  $0.1 \times \Delta_n^{-0.5}$  and

$0.5 \times \Delta_n^{-0.5}$ , corresponding to a window that is typically a couple of hours, as used by Ait-Sahalia and Xiu (2014) and Li and Xiu (2015). Since we use nonoverlapping windows, we try to choose one among the divisors of the total number of observations within the desired ranges of  $k_n$ 's. This helps improve the finite sample performance. For the PRL, the size of the smaller block  $l_n$  is around  $5 \times \Delta_n^{-0.5}$ , whereas the rule-of-thumb range of the larger block size  $k_n$  is approximately between  $2 \times \Delta_n^{-0.7}$  and  $10 \times \Delta_n^{-0.7}$ .

In Figure 1, we provide histograms based on 1000 repetitions of the IRL estimator using 1 min returns spanning a 5 day window in the Heston and LogV models. In both cases considered, the IRL estimator can recover the integrated leverage precisely, and its finite sample distribution is close to the asymptotic distribution.

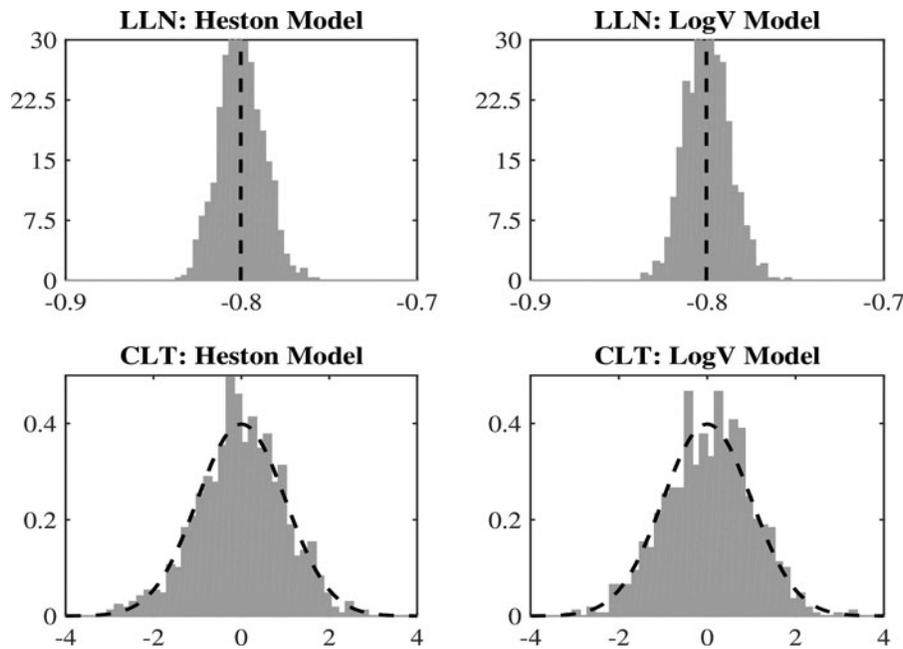
Table 1 provides a robustness check for the performance of the IRL estimator for different frequencies, time spans, and bandwidths ( $k_n$ s). In all cases, the IRL estimator performs well, with the best performance in the case of 1 week, 1 min data, which has the largest number of observations among the scenarios in Table 1.

Finally, we investigate the performance of both leverage estimators using a multi-factor volatility model. Under the objective measure, we have

$$\begin{aligned} dX_t &= (\mu_0 + \mu_1\sigma_t^2)dt + \sigma_t dW_t + J_X dN_t - \lambda_t \mu_X dt, \\ d\rho_t^2 &= \kappa_\rho(\bar{\rho} - \rho_t)dt + \gamma_\rho \sqrt{1 - \rho_t^2} dB_t, \\ d\sigma_t^2 &= (\eta + \kappa(\xi_t^2 - \sigma_t^2))dt + \gamma\sigma_t d\tilde{W}_t, \\ d\xi_t^2 &= \kappa_\xi(\bar{\xi}^2 - \xi_t^2)dt + \gamma_\xi \xi_t d\tilde{B}_t, \end{aligned}$$

where  $W_t$  and  $\tilde{W}_t$  are two Brownian motions with correlation  $\rho_t$ ,  $B_t$  and  $\tilde{B}_t$  are two additional independent Brownian motions,  $N_t$  is a Poisson process with state-dependent intensity  $\lambda_t = \lambda_0 + \lambda_1\sigma_t^2$ ,  $J_X$  is a random jump size of  $X$  satisfying  $J_X \sim N(\mu_X, \sigma_X^2)$ . We choose parameters as follows:  $\kappa = 5$ ,  $\eta = 0.5$ ,  $\kappa_\xi = 2$ ,  $\kappa_\rho = 4$ ,  $\gamma = 0.35$ ,  $\gamma_\xi = 0.3$ ,  $\gamma_\rho = 0.2$ ,  $\bar{\rho} = -0.8$ ,  $\bar{\xi}^2 = 0.06$ ,  $\mu_X = 0$ ,  $\sigma_X = 0.05$ ,  $\lambda_0 = 15$ ,  $\lambda_1 = 60$ ,  $\mu_0 = 0.05$ , and  $\mu_1 = 0.5$ . If the risk-neutral dynamics follows the same model, then  $\text{VIX}_t^2 = a + b\sigma_t^2 + c\xi_t^2$ , where the constants  $a$ ,  $b$ , and  $c$  depend on the parameters of the risk-neutral dynamics of  $X$ . We set them to  $\text{VIX}_t^2 = 100^2 \cdot (0.3 + 0.75\sigma_t^2 + 0.15\xi_t^2)$ , to match the observed empirical data.

In this multi-facto volatility model, Assumption 2 is violated. Therefore, our results for the IRL estimator do not apply, while the results for the PRL estimator continue to hold. Table 2 and Figure 2 demonstrate the behavior of the two estimators in this model. Figure 2 shows that IRL is biased (top left plot); the finite sample distribution is therefore shifted away from the asymptotic distribution (bottom left plot). On the other hand, the finite sample distribution of the PRL estimator is very close to the asymptotic distribution. The bias of the PRL estimator is small relative to the standard deviation (top right plot), but it has clearly much larger variability than the IRL estimator. Table 2 considers multiple settings and choices of the bandwidth. The absolute bias of the IRL estimator is sometimes smaller and sometimes larger than that of the PRL estimator. However, due to the much smaller variability of the IRL estimator, it has a



**Figure 1.** Simulation results: The IRL estimator. Note: The upper panel provides the histograms of the IRL, whereas the bottom panel plots the standardized histograms. The solid lines denote  $\bar{\rho}$  or the standard normal density. The parameters are  $\Delta_n = 1$  min,  $T = 5$  days, and  $k_n = 78$ .

much smaller root mean squared error (RMSE) in all settings considered.

The choice between the IRL and the PRL estimator therefore depends on the exact loss function of the researcher. The PRL estimator does not rely on Assumption 2, hence it is valid in a much larger class of models. The cost is larger variability of the resulting estimates compared to the IRL.

## 7. Empirical Results

### 7.1. Data and Preliminary Analysis

We consider an empirical application that uses the IRL estimator. We use intraday time series of E-mini S&P 500 futures and the VIX from the Tick Data Inc. The VIX sample period starts from September 22, 2003, when the CBOE began disseminating

prices for the VIX with the new methodology, and ends on December 31, 2013. We extend the intraday VIX series to January 1, 2003, by calculating the VIX using intraday options, based on the method developed by the CBOE, so that the full sample period covers 11 years in total. (See the CBOE White article on VIX at <http://www.cboe.com/micro/vix/vixwhite.pdf>.) We obtain a time series of the E-mini S&P 500 future prices by rolling over the front contracts. After removing nontrading days or half-trading days, our sample contains 2769 days. Overnight returns are excluded from our data. Figure 3 plots the time series of the intraday E-mini S&P 500 futures and the VIX from January 1, 2003 to December 31, 2013. After investigation of the signature plots of the IRL, we choose the sampling frequency of every 30 min for further time series analysis.

**Table 1.** Simulation results: The IRL estimator.

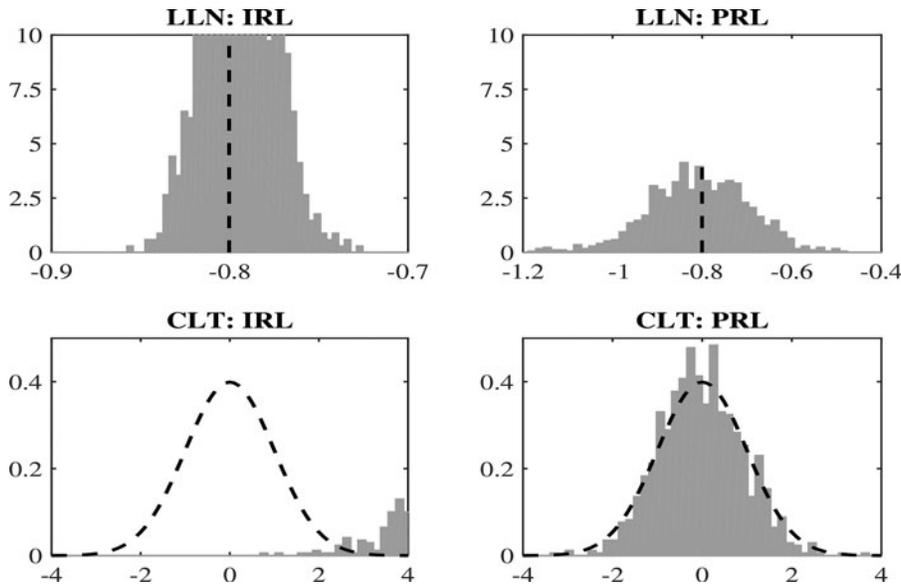
	Heston: 1 week, 1 min			LogV: 1 week, 1 min		
	bias	0.000	0.000	0.000	0.001	0.001
RMSE	0.018	0.015	0.017	0.026	0.026	0.026
	Heston: 1 week, 5 min			LogV: 1 week, 5 min		
	bias	0.002	0.002	0.002	0.004	0.004
RMSE	0.045	0.045	0.045	0.067	0.068	0.067
	Heston: 1 month, 5 min			LogV: 1 month, 5 min		
	bias	0.002	0.002	0.002	0.003	0.003
RMSE	0.048	0.047	0.048	0.052	0.052	0.052
	Heston: 1 month, 30 min			LogV: 1 month, 30 min		
	bias	0.025	0.026	0.030	0.025	0.026
RMSE	0.161	0.164	0.178	0.161	0.164	0.178

NOTE: Rows “bias” and “IQR” contain the average and interquartile range across simulations of the estimation error  $IRL_t - IL_t$ . For a given model, the three columns correspond to different bandwidths  $k_n$ , which are as follows: 39, 79, and 117 for 1 min sampling; 26, 39, and 78 for 5 min sampling; 13, 26, and 39 for 30 min sampling.

**Table 2.** Simulation results: The multi-factor volatility model.

	PRL: 1 month, 1 sec			IRL: 1 month, 1 sec		
	bias	0.002	-0.009	-0.001	0.009	0.009
RMSE	1.472	0.174	0.194	0.097	0.097	0.096
% outside [-1,1]	6.300	11.700	11.000	0.000	0.000	0.000
	PRL: 2 years, 1 min			IRL: 2 years, 1 min		
	bias	-0.013	0.012	0.022	0.005	0.005
RMSE	0.069	0.157	0.187	0.074	0.073	0.073
% outside [-1,1]	2.000	0.500	0.100	0.000	0.000	0.000
	PRL: 5 years, 5 min			IRL: 5 years, 5 min		
	bias	-0.036	0.034	0.049	0.007	0.007
RMSE	1.304	0.320	0.276	0.082	0.081	0.081
% outside [-1,1]	2.500	0.900	0.300	0.000	0.000	0.000

NOTE: For a given time span and frequency, the three columns correspond to different bandwidths. For the IRL, they are:  $k_n = 150, 3120,$  and  $6240$  for 1 sec sampling,  $k_n = 39, 78,$  and  $117$  for 1 min sampling, and  $k_n = 26, 39,$  and  $78$  for 5 min sampling. Choice of parameters for the PRL: for 1 sec data, the small block length is 1/2 day, the long subsamples are 3, 4, and 5 days; for 1 min data, the small block is 3 days, and the large blocks are 2, 3, and 4 months; for 5 min data, the small block is 7 days, and the large blocks are 4, 5, and 6 months.



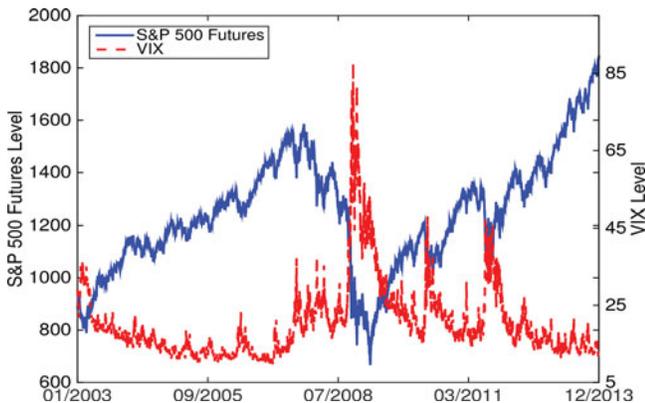
**Figure 2.** Simulation results: the multi-factor volatility model. Note: The top panel provides the histograms of the IRL and PRL estimators, whereas the right panel plots the standardized histograms. The gray areas show the histograms of the PRL estimators. The solid lines denote  $\bar{\rho}$  or the standard normal density. The parameters are  $\Delta_n = 1$  min and  $T = 2$  years. For the PRL, the small block (with  $l_n$  observations) is three days long, and the large block (with  $k_n$  observations) is two months long. For the IRL,  $k_n = 78$ .

### 7.2. Time Series Analysis

The current section implements a time series analysis of the leverage effect of the S&P500 index. We use the IRL estimator. The PRL estimator is noisy over short time intervals such as 1 month and hence not very informative about the variation of the true leverage effect across time. We use the VIX index as a volatility instrument.

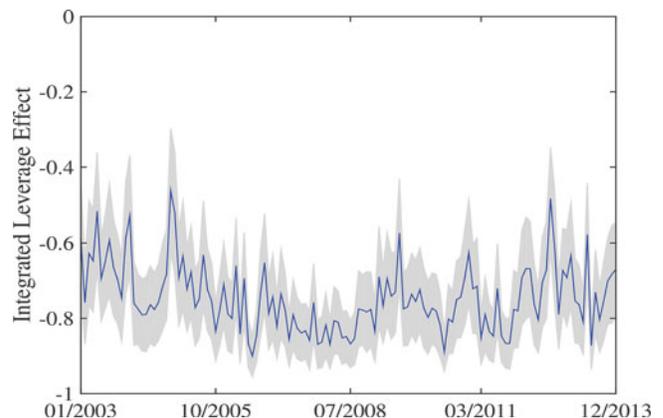
We first plot the monthly time series based on the IRL estimator in Figure 4, calculated using 30 min frequency. The average for the VIX-based IRL is  $-0.745$ . Both the time series pattern and the average leverage effect are not sensitive to the choice of  $\theta$  and sampling frequencies (15 min and lower).

The time series plot suggests two observations. First, our estimates indicate that the Brownian co-movement between the S&P 500 and its volatility is very pronounced and cannot be ignored. This challenges the pure jump specification of volatility process as suggested by Todorov and Tauchen (2011). Second, the correlation between the driving Brownian motions of the price and the volatility is clearly negative and time-varying.



**Figure 3.** Time series of the E-mini S&P 500 future prices and the VIX. Note: This figure plots the time series of the intraday E-mini S&P 500 futures and the VIX from January 1, 2003, to December 31, 2013.

It is interesting to explore if the documented time variation in the leverage effect estimates is related to variation in financial variables. For this purpose, we conduct the following regression analysis. First, we select the following economic variables: a measure of the credit risk, the default spread (DEF), calculated as the monthly difference between Moody’s Seasoned BAA and AAA corporate bond yields from the FRED; an illiquidity measure (ILLIQ), which is constructed by a monthly value-weighted firm-wise Amihud measure using CRSP data (Amihud 2002); a crisis dummy variable (NBER, 1 = crisis), constructed according to the NBER’s Business Cycle Dating Committee; and the logarithm of the monthly total debt-to-total-equity ratio (DER) of the S&P 500 index, downloaded from Bloomberg. (The debt-to-equity ratio for S&P 500 index is construed by Bloomberg, as the sum of short-term and long-term borrowing divided by the total shareholder’s equity, where the latter is equal to the sum of preferred equity, minority interest, and total common equity. We have also constructed the TED spread, the difference



**Figure 4.** Time series of leverage effect estimates. Note: This figure plots the monthly integrated leverage effect estimates of the E-mini S&P 500 futures for the years 2003–2013 using 30 min data, together with the uniform confidence intervals in gray.

**Table 3.** Time series regression results.

Variable	(1)	(2)	(3)	(4)	(5)	(6)
DEF		-0.066* (0.034)				0.019 (0.047)
ILLIQ			-0.139** (0.061)			-0.145** (0.056)
DER				-0.230* (0.126)		-0.329** (0.131)
NBER					-0.030** (0.014)	-0.031** (0.015)
AR	0.460*** (0.084)	0.439*** (0.085)	0.448*** (0.080)	0.464*** (0.083)	0.427*** (0.084)	0.426*** (0.078)
adj. R <sup>2</sup>	0.209	0.215	0.248	0.209	0.217	0.251

NOTE: Column (6) contains the results of the time series regression (14) with the VIX-based IRL as the dependent variable. Columns (1)–(5) contain results of the corresponding simple regressions. DER denotes the log total debt-to-total-equity ratio of the S&P 500 index, DEF denotes the Default spread, ILLIQ is a monthly illiquidity measure, and NBER denotes the crisis dummy. The reported standard errors are based on the Newey–West procedure. All regressions include an intercept, which is not reported. \*\*\*, \*\*, \* denote significance at 1%, 5% and 10% levels.

between the three-month LIBOR and the three-month T-Bill interest rate obtained from the FRED, a liquidity measure from Pastor and Stambaugh (2003), as well as an equally weighted firm-wise Amihud illiquidity measure. An earlier draft includes the results based on these measures. The economic interpretation of the regression results is robust to the choice of different variables.) Our sample consists of monthly leverage effects over 11 years, which is a total of 132 observations.

We estimate the following AR(1) regression:

$$\begin{aligned} \text{LEV}_t = & \beta_0 + \beta_1 \varepsilon_{\text{DEF},t} + \beta_2 \varepsilon_{\text{ILLIQ},t} + \beta_3 \varepsilon_{\text{DER},t} \\ & + \beta_4 \text{NBER}_t + \beta_5 \text{LEV}_{t-1} + \varepsilon_t, \end{aligned} \quad (14)$$

where  $\varepsilon_{\cdot,t}$  denotes the corresponding AR(1) innovation of each covariate. From the regression results in Table 3, we find that the credit and liquidity factors are relevant to the leverage effect and their coefficients have signs consistent with the economic intuition (recall the dependent variable is negative). They imply that the leverage effect is magnified in bad times, that is, a 1 percentage drop of stock price when credit risk is high and liquidity is low, may lead to a larger percentage increase of risk. The same conclusion holds with NBER dummy, that is, crisis periods display a larger leverage effect. Moreover, the debt-to-equity ratio is significant, which supports the financial leverage hypothesis of Black (1976) that the debt-to-equity ratio is one of the determinants of the leverage effect. The latter finding is robust with respect to various alternative specifications, which are omitted for brevity.

### 7.3. Specification Test

We now implement the Durbin–Wu–Hausman test of Section 5. The test statistic is given in Equation (13). We use 30 min observations of E-mini S&P 500 future prices and the VIX index observations at the same frequency. The PRL estimator is  $-0.634$  (with the standard error 0.073), while the IRL estimator is  $-0.745$ . The  $p$ -value of the two-sided test is 0.13, so we do not reject the null hypothesis of PRL and IRL yielding the same estimates in large samples.

## 8. Conclusion

We propose two nonparametric estimators of the leverage effect. The first estimator, PRL, only uses the price data, and corrects for biases that arise due to the preliminary estimation of volatility. Our proof of the asymptotic distribution of the PRL extends several key methods in the literature. Our second estimator, IRL, uses the data from two sources, the stock price as well as a volatility instrument such as Black–Scholes implied volatility or the VIX. We provide the asymptotic theory for the IRL estimator as well. The two estimators we develop are complementary to each other and have their own advantages and disadvantages. The PRL estimator is valid in a very general class of models, while the IRL estimator has a faster rate of convergence. Empirically, we find the PRL estimator has much larger standard errors than the IRL estimator, but is nevertheless useful for estimating the integrated leverage effect over the span of several years. We conduct a time-series study with the IRL, and find significant relationship between the leverage effect and the debt-to-equity ratio, which supports the leverage hypothesis of Black (1976).

## Supplementary Materials

Appendix A contains the proof of Theorem 1 of the main text. Appendix B contains the proof of Theorem 2 of the main text. Appendix C contains the proof of Lemma A2, which is stated in Appendix A. Appendix D contains the proof of Lemma A3, which is stated in Appendix A.

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